Analytical solution of magnetohydrodynamics generalized Burger's fluid embedded with porosity

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ABSTRACT

This analysis is devoted to investigate analytical solutions of magnetohydrodynamics (MHD) generalized Burger's fluid embedded with porous medium as a sum of Newtonian and non-Newtonian forms. The solutions are investigated for velocity field and shear stress and governing partial differential equations have been solved via the integral transforms. The solutions for velocity field and shear stress have been expressed into compact form i-e in terms of series form. The general solutions also satisfy initial and boundary condition and particularized for special cases along with sum of Newtonian and non-Newtonian forms. The impacts of permeability (porosity), magnetism and several rheological parameters have been analyzed for fluid flows by portraying graphical illustrations. The graphs are depicted via latest software namely Mathematica and Mathcad (16) packages.

1. Introduction

The linearized relationship between rate of strain and stress cannot be characterized by non-Newtonian fluids because of the involvement of stress tensor and relaxation which convert viscoelastic fluids into highly complicated and nonlinear in contrast with Newtonian fluids. Non-Newtonian fluids even have gotten considerable attention because of their useful applications in industries and engineering. For instance, emulsion, heavy oils, chyme, blood and polymer solutions etc. Such fluids are modeled by their proposed constitutive relationships with symmetries. The description of these fluids are modeled to characterize viscoelastic behavior of flows as Maxwell, Oldrord-B, Burger models characterize small relaxation phenomenon, retardation times, rheological properties respectively (Zhaosheng and Jianzhong, 1998; Rajagopal, 1982; Zhang et al., 2007; Tan and Masuoka 2007; Abro, 2016; Fetecau et al., 2008; Abro and Shaikh, 2015). Furthermore, the main significance of Burger's model is to describe the cheese in food products and asphalts in geomechanics. Also, the characteristics of motion of

the earths' mantle, asphalt concrete, asphalt, Olivine rocks, geological structures and propagation of seismic waves in the interior of the earth can be described by this model (Murali Krishnan and Rajagopal, 2004). Due to diverse applications, Burgers model has received much attention. In continuation we present few references for the study of Burger model (Jamil, 2012; Tong, 2010; Jamil and Fetecau, 2010; Khan et al., 2006; Hayat et al., 2010). The analysis for electrical conducting fluid flows is termed as magnetohydrodynamics has diverted attentions of many scientists and researchers due to its vast application in science and engineering. In geophysics and astrophysics, magnetohydrodynamics is useful to study solar plasma, terrestrial cores, cosmical fluid dynamics, structure of rotating magnetic stars, and stellar structures. The significance of magnetohydrodynamics in science and engineering, particularly in industrial applications of magnetohydrodynamics lies in extraction of geothermal energy, crystal growth in the field of semiconductors, magnetohydrodynamics (MHD) pumps, metallurgical and material processes and nuclear reactors (Hsiao, 2011; Zheng et al., 2012; Ahmad and Nazar, 2010). Obtaining the analytical solutions in presence of porous medium and magnetohydrodynamics is not an easy task, that why solutions in literatures are very rare. Having these complexities several scholars are busy in order to contribute in this phenomenon (presence and

absence of porous medium and MHD). In our cognizance, some valuable contribution has been made by Khan et al. (2011) in which they studied magnetohydrodynamics flow of second grade fluid in porous medium for closed form solution. Hussain et al. (2010) has investigated second grade fluid in porous medium under oscillatory flows. Khan et al. (2012) has analyzed effects of thermal diffusion and ramped wall temperature for magnetohydrodynamics free convection flow in a porous medium. Furthermore few interested studies especially on MHD have been made by several researcher, we include here references in (Hayat et al., 2008a, 2008b; Abbasbandy et al., 2014; Freidoonimehr et al., 2015; Rashidi and Erfani, 2012; Rashidi et al., 2014). By the inspirations and motivations of above mentioned work, this analysis is devoted to investigate analytical solutions of MHD generalized Burger's fluid embedded with porous medium as a sum of Newtonian and non-Newtonian forms. The solutions are investigated for velocity field and shear stress and governing partial differential equations have been solved via the integral transformational techniques. The solutions for velocity field and shear stress have been expressed into compact form i-e in terms of series form. The general solutions also satisfy initial and boundary condition and particularized for special cases along with sum of Newtonian and non-Newtonian forms. The impacts of various rheological parameters have been analyzed for six models namely (i) Generalized Burger model, (ii) Burger model, (iii) Oldroyd-B model, (iv) Maxwell model, (v) Second Grade model and (vi) Newtonian model. These models are also discussed with and without porous medium and magnetohydrodynamics effects on fluid flow.

2. Modeling of governing equations

The constitutive equations for an incompressible generalized Burgers' fluid are (Jamil, 2012; Tong, 2010; Jamil and Fetecau, 2010; Khan et al., 2006) (Eq. 1)

$$T = -p\mathrm{I} + S, \quad \mu\left(A_1 + \lambda_4 \frac{\delta^2 A_1}{\delta t^2} + \lambda_3 \frac{\delta A_1}{\delta t}\right) = S + \lambda_2 \frac{\delta^2 S}{\delta t^2} + \lambda_1 \frac{\delta S}{\delta t}, \tag{1}$$

where, p the pressure, I the identity tensor, S is the extra-stress tensor, μ the dynamic viscosity, $A = L + L^T$ is the first Rivlin-Ericksen tensor, L denotes velocity gradient, λ_1 and λ_3 ($< \lambda_1$) are relaxation and retardation time, λ_2 and λ_4 are new material parameters of the generalized Burgers' fluid and $\frac{\delta}{\delta t}$ denotes the upper convected time derivative defined as (Eq. 2)

$$\frac{\delta^2 S}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta S}{\delta t}\right), \quad \frac{\delta S}{\delta t} = \frac{dS}{dt} - LS - SL^T, \tag{2}$$

$\frac{d}{dt}$ represents material time derivative. The unsteady flow of incompressible fluid is governed by (Eqs. 3 and 4):

$$\mathrm{div}\,\boldsymbol{V} = 0, \tag{3}$$
$$\rho\frac{dV}{dt} = \mathrm{div}\,T - \sigma B_0^2 V + R, \tag{4}$$

where, ρ is the density of the fluid, V is the velocity, σ the electrical conductivity of the fluid, B_0 the applied magnetic field's magnitude and R denotes the Darcy's resistance. For the problem under consideration we assume a velocity field an extra-stress tensor of the form (Eq. 5)

$$V = V(y,t) = w(y,t)i, S = S(y,t), \tag{5}$$

where i is the unit vector along the x-co-ordinate direction. Introducing Eq. 5 in Eq. 1 and keeping initial conditions in mind (Eq. 6)

$$S(y,0) = \frac{\partial S(y,0)}{\partial t} = 0, \tag{6}$$

yields $\tau_{zz} = \tau_{xz} = \tau_{yy} = \tau_{yz} = 0$, and (Eq. 7)

$$\mu\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)\frac{\partial w}{\partial y} = \left(\lambda_2 \frac{\partial^2}{\partial t^2} + \lambda_1 \frac{\partial}{\partial t} + 1\right)S_{xy}, \tag{7}$$

in which tangential stress is S_{xy}. With reference (Tan and Masuoka, 2007), the generalized Burger's fluid has relation for R is

$$-\frac{\mu\emptyset}{k}\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)\boldsymbol{V}(y,t) = \boldsymbol{R}\left(\lambda_2 \frac{\partial^2}{\partial t^2} + \lambda_1 \frac{\partial}{\partial t} + 1\right), \tag{8}$$

where, k is the permeability of the porous medium and $\emptyset$ is the porosity. Assuming that there is no pressure gradient in the flow direction and introducing Eq. 5 into Eq. 4 keeping in mind Eqs. 7, 8, we obtain following governing equations as (Jamil, 2012; Tong, 2010; Jamil and Fetecau, 2010; Khan et al., 2006; Hayat et al., 2010) (Eqs. 9 and 10)

$$\left(\lambda_2 \frac{\partial^2}{\partial t^2} + \lambda_1 \frac{\partial}{\partial t} + 1\right)\frac{\partial w(y,t)}{\partial t} = \nu\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)\frac{\partial^2 w(y,t)}{\partial y^2} - B\left(\lambda_2 \frac{\partial^2}{\partial t^2} + \lambda_1 \frac{\partial}{\partial t} + 1\right)w(y,t) - \Phi\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)w(y,t), \tag{9}$$

$$\left(\lambda_2 \frac{\partial^2}{\partial t^2} + \lambda_1 \frac{\partial}{\partial t} + 1\right)\tau(y,t) = \mu\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)\frac{\partial w(y,t)}{\partial t}, \tag{10}$$

where, $= \frac{\mu}{\rho}$, $B = \frac{\sigma B_0}{\rho}$, $\Phi = \frac{\mu\emptyset}{k}$ are kinematic viscosity, applied magnetic field, porous medium of the fluid respectively.

3. Statement of the problem

We consider an incompressible and electrically conducting generalized Burgers fluid in porous medium occupying the space above a plate perpendicular to the y-axis. The plate is permeated under an influence of magnetic field B_0 normal to the flow in porous medium. For $t = 0^+$ the plate begins to accelerate in its own plane with velocity $w(0,t) = UH(t)t^p$. Due to the shear, the fluid above the plate

is gradually moved, while the governing equations are given by Eqs. 9 and 10 and corresponding initial, boundary and natural conditions are (Eqs. 11-13):

$$\frac{\partial^2 w(y,0)}{\partial t^2} = \frac{\partial w(y,0)}{\partial t} = w(y,0) = 0, \qquad y > 0, \qquad (11)$$

$$w(0,t) = UH(t)t^p \qquad\qquad\qquad t \geq 0. \qquad (12)$$

$$w(y,t), \frac{\partial w(y,t)}{\partial t} \to 0 \quad as \quad y \to \infty \quad and \quad t > 0, \qquad (13)$$

4. Solution of the problem

4.1. Calculation of velocity field

In order to explore the exact solution, we shall apply the Fourier sine transform to Eq. 9, and having in account the imposed conditions (11) and (12), we find that(Eq. 14)

$$\left(\lambda_2 \frac{\partial^2}{\partial t^2} + \lambda_1 \frac{\partial}{\partial t} + 1\right)\frac{\partial w_s(\xi,t)}{\partial t} - \nu\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)\left\{-\xi^2 w_s(\xi,t) + \sqrt{\frac{2}{\pi}} w(0,t)\right\} + B\left(\lambda_2 \frac{\partial^2}{\partial t^2} + \lambda_1 \frac{\partial}{\partial t} + 1\right)w_s(\xi,t) + \Phi\left(\lambda_4 \frac{\partial^2}{\partial t^2} + \lambda_3 \frac{\partial}{\partial t} + 1\right)w_s(\xi,t) = 0, \qquad (14)$$

where, $w_s(\xi,t)$ is the Fourier sine transform of $w(y,t)$ defined in Eq. 14 (Eq. 15)

$$w_s(\xi,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty w(y,t) Sin(y\xi)dy, \qquad (15)$$

where the Fourier sine transform has to justify imposed conditions Eq. 11, (Eq. 16)

$$\frac{\partial^2 w_s(\xi,0)}{\partial t^2} = \frac{\partial w_s(\xi,0)}{\partial t} = w_s(\xi,0) = 0, \qquad \xi > 0, \qquad (16)$$

applying Laplace transform to Eq. 15 and keeping imposed conditions Eqs. 11 and 12, we obtain

$$\bar{w}_s(\xi,s) = \sqrt{\frac{2}{\pi}} \frac{U\nu\xi p!}{s^{p+1}} \qquad (17)$$

$$\left[\frac{(\lambda_4 s^2 + \lambda_3 s + 1)}{[\lambda_2 s^3 + \{\lambda_1 + \lambda_2 B + \lambda_4(\Phi + \nu\xi^2)\}s^2 + \{1 + \lambda_1 B + \lambda_3(\Phi + \nu\xi^2)\}s + (\Phi + B + \nu\xi^2)]}\right]$$

in order to write the solutions as a sum of Newtonian and non-Newtonian Part $w_N(y,t) + w_{Non}(y,t)$, we make $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ in Eq. 17, we have expression for Newtonian part as (Eq. 18)

$$\bar{w}_{s(N)}(\xi,s) = \sqrt{\frac{2}{\pi}}\frac{U\nu\xi p!}{s^{p+1}(s+\Phi+B+\nu\xi^2)}, \qquad (18)$$

substituting Eq. 18 into Eq. 17, we find balanced equation (Eq. 19)

$$\bar{w}_s(\xi,s) = U\sqrt{\frac{2}{\pi}}\frac{\nu\xi p!}{(\Phi+B+\nu\xi^2)}\left\{\frac{1}{s^{p+1}} - \frac{1}{s^p(s+\Phi+B+\nu\xi^2)}\right\} + U\sqrt{\frac{2}{\pi}}\frac{\nu\xi p!}{\lambda_2} \times \left[\frac{(\lambda_4-\lambda_3)s^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s + (\lambda_3 B - \lambda_1)}{s^p(s+\Phi+B+\nu\xi^2)(s-s_1)(s-s_2)(s-s_3)}\right] \qquad (19)$$

where, $(s - s_1)(s - s_2)(s - s_3)$ are the roots of an algebraic equation

$$\lambda_2 s^3 + \{\lambda_1 + \lambda_2 B + \lambda_4(\Phi + \nu\xi^2)\}s^2 + \{1 + \lambda_1 B + \lambda_3(\Phi + \nu\xi^2)\}s + (\Phi + B + \nu\xi^2).$$

Inverting Eq. 19 by means of Fourier sine transform, we get (Eq. 20)

$$\bar{w}(y,s) = \frac{2U\nu\xi p!}{\pi}\int_0^\infty \frac{sin(y\xi)}{(\Phi+B+\nu\xi^2)}\left\{\frac{1}{s^{p+1}} - \frac{1}{s^p(s+\Phi+B+\nu\xi^2)}\right\}d\xi + \frac{2U\nu p!}{\pi\lambda_2}\int_0^\infty \xi sin(y\xi) \times \left[\frac{(\lambda_4-\lambda_3)s^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s + (\lambda_3 B - \lambda_1)}{s^p(s+\Phi+B+\nu\xi^2)(s-s_1)(s-s_2)(s-s_3)}\right]d\xi, \qquad (20)$$

employing the fact of integral Eq. 21

$$\int_0^\infty \frac{\xi sin(\gamma\xi)}{\alpha^2+\xi^2}d\xi = \frac{\pi}{2}e^{-\alpha\gamma}, \qquad \alpha > 0, \qquad (21)$$

and using inverse Laplace transform in Eq. 20, we obtain convolution product as (Eq. 22)

$$w(y,t) = UH(t)t^p Exp\left(-y\sqrt{\frac{\Phi+B}{\nu}}\right) - \frac{2UH(t)\nu p}{\pi}\int_0^\infty \frac{\xi sin(y\xi)}{(\Phi+B+\nu\xi^2)}t^{p-1} * Exp(-\Phi-B-\nu\xi^2)d\xi + \frac{2UH(t)\nu p}{\pi\lambda_2} \times \int_0^\infty \xi sin(y\xi)\, t^{p-1} *$$

$$\left[\begin{array}{l} \frac{(\lambda_4-\lambda_3)s_1^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s_1 + (\lambda_3 B-\lambda_1) Exp(s_1)t}{(s_1+\Phi+B+\nu\xi^2)(s_1-s_2)(s_1-s_3)} \\ + \frac{(\lambda_4-\lambda_3)s_2^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s_2 + (\lambda_3 B-\lambda_1) Exp(s_2)t}{(s_2+\Phi+B+\nu\xi^2)(s_2-s_1)(s_2-s_3)} \\ + \frac{(\lambda_4-\lambda_3)s_3^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s_3 + (\lambda_3 B-\lambda_1) Exp(s_3)t}{(s_3+\Phi+B+\nu\xi^2)(s_3-s_1)(s_3-s_2)} \\ + \frac{(\lambda_4-\lambda_3)(\Phi+B+\nu\xi^2)^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}(\Phi+B+\nu\xi^2) + (\lambda_3 B-\lambda_1)}{(s_1+\Phi+B+\nu\xi^2)(s_2+\Phi+B+\nu\xi^2)(s_3+\Phi+B+\nu\xi^2)} \\ \times Exp(-\Phi-B-\nu\xi^2)t \end{array}\right] d\xi, \qquad (22)$$

finally velocity field is expressed in compact form in terms of series, (Eq. 23)

$$w(y,t) = w_N(y,t) + \frac{2UH(t)\nu p}{\pi\lambda_2}\int_0^\infty \int_0^t \xi sin(y\xi)(t-z)^{p-1} \times$$

$$\left[\begin{array}{l} \frac{(\lambda_4-\lambda_3)(\Phi+B+\nu\xi^2)^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}(\Phi+B+\nu\xi^2) + (\lambda_3 B-\lambda_1)}{(s_1+\Phi+B+\nu\xi^2)(s_2+\Phi+B+\nu\xi^2)(s_3+\Phi+B+\nu\xi^2)} \\ \times e^{(-\Phi-B-\nu\xi^2)z} + \sum_{n=1}^3 \frac{(\lambda_4-\lambda_3)s_n^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s_n + (\lambda_3 B-\lambda_1)e^{s_n z}}{(s_n+\Phi+B+\nu\xi^2)(s_n-s_{(3-n)!})(s_n-s_{4-(n-1)!})} \end{array}\right] d\xi\, dz.$$

$$(23)$$

where, $w_N(y,t) = UH(t)t^p Exp\left(-y\sqrt{\frac{\Phi+B}{\nu}}\right) - \frac{2UH(t)\nu p}{\pi}\int_0^\infty \int_0^t \frac{\xi sin(y\xi)}{(\Phi+B+\nu\xi^2)}(t-z)^{p-1}Exp(-\Phi-B-\nu\xi^2)z\, d\xi\, dz$ is the Newtonian part velocity field.

4.2. Calculation of shear stress

For finding shear stress, we apply Laplace transform to Eq. 10, we find that (Eq. 24)

$$\bar{\tau}(y,s) = \mu\left(\frac{\lambda_4 q^2 + \lambda_3 q + 1}{\lambda_2 q^2 + \lambda_1 q + 1}\right)\frac{\partial \bar{w}(y,s)}{\partial y}, \tag{24}$$

in order to write the solutions as a sum of Newtonian and non-Newtonian Part $\tau_N(y,t) + \tau_{Non}(y,t)$, we make $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ in Eq. 24, we have expression for Newtonian part as (Eq. 25)

$$\bar{\tau}_{(N)}(y,s) = \mu\frac{\partial \bar{w}(y,s)}{\partial y}, \tag{25}$$

$$\bar{\tau}(y,s) = -\frac{2U\,\mu\,v\,p!}{\pi}\int_0^\infty \frac{\xi^2 cos(y\xi)}{(\Phi+B+v\xi^2)}\frac{1}{s^p(s+\Phi+B+v\xi^2)}d\xi - \frac{2U\mu vp!}{\pi\lambda_2}\int_0^\infty \frac{\xi^2 cos(y\xi)}{(\Phi+B+v\xi^2)}\frac{1}{s^p}\,d\xi \times$$

$$\left[\begin{array}{l} \frac{(\lambda_4-\lambda_2)s_1^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s_1 + (\lambda_3-\lambda_1)B}{(s_1+\Phi+B+v\xi^2)\,(s_1-s_2)\,(s_1-s_3)(s-s_1)} \\ + \frac{(\lambda_4-\lambda_2)s_1^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s_1 + (\lambda_3-\lambda_1)B}{(s_2+\Phi+B+v\xi^2)\,(s_2-s_1)\,(s_2-s_3)(s-s_2)} \\ + \frac{(\lambda_4-\lambda_2)s_1^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s_1 + (\lambda_3-\lambda_1)B}{(s_2+\Phi+B+v\xi^2)\,(s_3-s_1)\,(s_3-s_2)(s-s_3)} \\ + \frac{(\lambda_4-\lambda_2)(\Phi+B+v\xi^2)^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}(\Phi+B+v\xi^2)+(\lambda_3-\lambda_1)B}{(s_1+\Phi+B+v\xi^2)(s_2+\Phi+B+v\xi^2)(s_3+\Phi+B+v\xi^2)(s+\Phi+B+v\xi^2)}, \end{array}\right] \tag{27}$$

finally applying inverse Laplace transform to Eq. 27, we expressed shear stress in compact form in terms of series,

$$\tau(y,t) = \tau_N(y,t) - \frac{2U\,H(t)\,\mu\,v\,p}{\pi\lambda_2}\int_0^\infty\int_0^t \frac{\xi^2 cos(y\xi)}{(\Phi+B+v\xi^2)}(t-z)^{p-1}\times$$

$$\left[\begin{array}{l} \frac{(\lambda_4-\lambda_2)(\Phi+B+v\xi^2)^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}(\Phi+B+v\xi^2)+(\lambda_3-\lambda_1)B}{(s_1+\Phi+B+v\xi^2)(s_2+\Phi+B+v\xi^2)(s_3+\Phi+B+v\xi^2)} \\ \times e^{(-\Phi-B-v\xi^2)z} + \sum_{n=1}^3 \frac{(\lambda_4-\lambda_2)s_n^2 + \{(\lambda_4-\lambda_2)B+(\lambda_3-\lambda_1)\}s_n + (\lambda_3-\lambda_1)Be^{s_n t}}{(s_n+\Phi+B+v\xi^2)\,(s_n-s_{(3-n)!})\,(s_n-s_{4-(n-1)!})} \end{array}\right]d\xi\,dz. \tag{28}$$

where,

$$\tau_N(y,t) = -\frac{2U\,H(t)\,\mu\,v\,p}{\pi}\int_0^\infty\int_0^t \frac{\xi^2 cos(y\xi)}{(\Phi+B+v\xi^2)}(t-z)^{p-1}Exp(-\Phi-B-v\xi^2)\,z\,d\xi\,dz.$$

is the Newtonian part of shear stress.

5. Limiting case

Employing same procedure we obtain here some special cases of general solutions of velocity field and corresponding shear stress.

$$w_B(y,t) = w_N(y,t) - \frac{2U\,H(t)\,v\,p}{\pi\lambda_2}\int_0^\infty\int_0^t \xi\,sin(y\xi)\,(t-z)^{p-1}\times$$

substituting the expression of Eqs. 18 and 25 into Eq. 24, we find balanced equation (Eq. 26)

$$\bar{\tau}(y,s) = -\frac{2U\mu vp!}{\pi}\int_0^\infty \frac{\xi^2 cos(y\xi)}{(\Phi+B+v\xi^2)}\frac{1}{s^p(s+\Phi+B+v\xi^2)}d\xi +$$

$$\frac{2U\mu vp!}{\pi\lambda_2}\int_0^\infty \frac{\xi^2 cos(y\xi)}{(\Phi+B+v\xi^2)}\frac{1}{s^{p+1}}\times\left[\frac{1}{(s+\Phi+B+v\xi^2)} - \frac{\lambda_4 s^2 + \lambda_3 s + 1}{(s-s_1)(s-s_2)(s-s_3)}\right]d\xi, \tag{26}$$

now, writing Eq. 26 for more suitable following equivalent form (Eq. 27)

5.1. Solutions of burger's fluid

Letting $\lambda_4 = 0$ in Eqs. 23 and 28, we obtain the velocity field and shear stress of Burger's fluid

$$\left[\begin{array}{l} \frac{\lambda_3 s_4^2 + (\lambda_3-B\lambda_2-\lambda_1)s_4 + (\lambda_3 B-\lambda_1)e^{s_4 z}}{(s_4+\Phi+B+v\xi^2)\,(s_4-s_5)\,(s_4-s_6)} \\ + \frac{\lambda_3 s_5^2 + (\lambda_3-B\lambda_2-\lambda_1)s_5 + (\lambda_3 B-\lambda_1)e^{s_5 z}}{(s_5+\Phi+B+v\xi^2)\,(s_5-s_4)\,(s_5-s_6)} \\ + \frac{\lambda_3 s_6^2 + (\lambda_3-B\lambda_2-\lambda_1)s_6 + (\lambda_3 B-\lambda_1)e^{s_6 z}}{(s_6+\Phi+B+v\xi^2)\,(s_6-s_4)\,(s_6-s_5)} \\ \frac{\lambda_3(\Phi+B+v\xi^2)^2 - \{\lambda_3-\lambda_2 B-\lambda_1\}(\Phi+B+v\xi^2)+(\lambda_3 B-\lambda_1)}{(s_1+\Phi+B+v\xi^2)(s_2+\Phi+B+v\xi^2)(s_3+\Phi+B+v\xi^2)}e^{(-\Phi-B-v\xi^2)z} \end{array}\right]d\xi\,dz, \tag{29}$$

and corresponding shear stress

$$\tau(y,t) = \tau_N(y,t) - \frac{2U\,H(t)\,\mu\,v\,p}{\pi\lambda_2}\int_0^\infty\int_0^t \frac{\xi^2 cos(y\xi)}{(\Phi+B+v\xi^2)}(t-z)^{p-1}\times$$

$$\left[\begin{array}{l} \frac{\lambda_2 s_4^2 - (\lambda_1+\lambda_2 B-\lambda_3)s_4 + (\lambda_3-\lambda_1)B\,Exp(s_4 z)}{(s_4+\Phi+B+v\xi^2)(s_4-s_5)(s_4-s_6)} \\ + \frac{\lambda_2 s_5^2 - (\lambda_1+\lambda_2 B-\lambda_3)s_5 + (\lambda_3-\lambda_1)B\,Exp(s_5 z)}{(s_5+\Phi+B+v\xi^2)(s_5-s_4)(s_5-s_6)} \\ + \frac{\lambda_2 s_6^2 - (\lambda_1+\lambda_2 B-\lambda_3)s_6 + (\lambda_3-\lambda_1)B\,Exp(s_6 z)}{(s_6+\Phi+B+v\xi^2)(s_6-s_4)(s_6-s_5)} \\ + \frac{\lambda_2(\Phi+B+v\xi^2)^2 - (\lambda_3-\lambda_2 B+\lambda_1)(\Phi+B+v\xi^2)+(\lambda_3-\lambda_1)B}{(s_4+\Phi+B+v\xi^2)(s_5+\Phi+B+v\xi^2)(s_6+\Phi+B+v\xi^2)}e^{(-\Phi-B-v\xi^2)z} \end{array}\right]d\xi\,dz \tag{30}$$

$(s - s_4)(s - s_5)(s - s_6)$ are the roots of an algebraic equation:

$$\lambda_2 s^3 + (\lambda_1 + \lambda_2 B)s^2 + \{1 + \lambda_1 B + \lambda_3(\Phi + \nu\xi^2)\}s + (\Phi + B + \nu\xi^2).$$

5.2. Solutions of Oldroyd-B fluid

Letting $\lambda_4 = \lambda_2 = 0$ in Eqs. 23 and 28, we obtain the velocity field and shear stress of Oldrord-B fluid

$$w_{OB}(y,t) = w_N(y,t) + \frac{2\,U\,H(t)\,\nu\,p}{\pi\,\lambda_1}\int_0^\infty \int_0^t \xi \sin(y\xi)\ (t-z)^{p-1} \times$$

$$\left[\begin{array}{l}\frac{(\lambda_1-\lambda_3)(s_7+B)e^{s_7 z}}{(s_7+\Phi+B+\nu\xi^2)(s_7-s_8)} + \frac{(\lambda_1-\lambda_3)(s_8+B)e^{s_8 z}}{(s_8+\Phi+B+\nu\xi^2)(s_8-s_7)} \\ + \frac{(\lambda_1-\lambda_3)(1+\Phi+B+\nu\xi^2)}{(s_7+\Phi+B+\nu\xi^2)(s_8+\Phi+B+\nu\xi^2)}e^{(-\Phi-B-\nu\xi^2)z}\end{array}\right] d\xi\ dz,$$

(31)

and corresponding shear stress

$$\tau_{OB}(y,t) = \tau_N(y,t) - \frac{2\,U\,H(t)\,\mu\,\nu\,p}{\pi\,\lambda_1}\int_0^\infty \int_0^t \frac{\xi^2 \cos(y\xi)}{(\Phi+B+\nu\xi^2)}\ (t-z)^{p-1} \times$$

$$\left[\begin{array}{l}\frac{(\lambda_1-\lambda_3)(s_7+\lambda_3\nu\xi^2+B)e^{s_7 z}}{(s_7+\Phi+B+\nu\xi^2)(s_7-s_8)} + \frac{(\lambda_1-\lambda_3)(s_8+\lambda_3\nu\xi^2+B)e^{s_8 z}}{(s_8+\Phi+B+\nu\xi^2)(s_8-s_7)} \\ + \frac{(\lambda_1-\lambda_3)(\lambda_3\nu\xi^2+\Phi+2B+\nu\xi^2)}{(s_7+\Phi+B+\nu\xi^2)(s_8+\Phi+B+\nu\xi^2)}e^{(-\Phi-B-\nu\xi^2)z}\end{array}\right] d\xi\ dz,$$

(32)

$(s - s_7)(s - s_8)$ are the roots of an algebraic equation:

$$\lambda_1 s^2 + \{1 + \lambda_1 B + \lambda_3(\Phi + \nu\xi^2)\}s + (\Phi + B + \nu\xi^2).$$

5.3. Solutions of Maxwell fluid

Letting $\lambda_4 = \lambda_3 = \lambda_2 = 0$ in Eqs. 23 and 28, we obtain the velocity field and shear stress of Maxwell fluid

$$w_M(y,t) = w_N(y,t) + \frac{2\,U\,H(t)\,\nu\,p}{\pi\,\lambda_1}\int_0^\infty \int_0^t \xi \sin(y\xi)\ (t-z)^{p-1} \times \left[\begin{array}{l}\frac{\lambda_1(s_9+B)e^{s_9 z}}{(s_9+\Phi+B+\nu\xi^2)(s_9-s_{10})} + \frac{\lambda_1(s_{10}+B)e^{s_{10} z}}{(s_{10}+\Phi+B+\nu\xi^2)(s_{10}-s_9)} \\ + \frac{\lambda_1(1+\Phi+B+\nu\xi^2)}{(s_9+\Phi+B+\nu\xi^2)(s_{10}+\Phi+B+\nu\xi^2)}e^{(-\Phi-B-\nu\xi^2)z}\end{array}\right] d\xi\ dz,$$

(33)

and corresponding shear stress

$$\tau_M(y,t) = \tau_N(y,t) - \frac{2\,U\,H(t)\,\mu\,\nu\,p}{\pi\,\lambda_1}\int_0^\infty \int_0^t \frac{\xi^2 \cos(y\xi)}{(\Phi+B+\nu\xi^2)}\ (t-z)^{p-1} \times \left[\begin{array}{l}\frac{\lambda_1(s_9+B)e^{s_9 z}}{(s_9+\Phi+B+\nu\xi^2)(s_9-s_{10})} + \frac{\lambda_1(s_{10}+\lambda_3\nu\xi^2+B)e^{s_{10} z}}{(s_{10}+\Phi+B+\nu\xi^2)(s_{10}-s_9)} \\ + \frac{\lambda_1(\Phi+2B+\nu\xi^2)}{(s_9+\Phi+B+\nu\xi^2)(s_{10}+\Phi+B+\nu\xi^2)}e^{(-\Phi-B-\nu\xi^2)z}\end{array}\right] d\xi\ dz,$$

(34)

$(s - s_9)(s - s_{10})$ are the roots of an algebraic equation:

$$\lambda_1 s^2 + (1 + \lambda_1 B)s + (\Phi + B + \nu\xi^2).$$

5.4. Solutions of second grade fluid

Letting $\lambda_4 = \lambda_2 = \lambda_1 = 0$ in Eqs. 23 and 28, we obtain the velocity field and shear stress of Second Grade fluid

$$w_{SG}(y,t) = w_N(y,t) + \frac{2\,U\,H(t)\,\nu\,p}{\pi}\int_0^\infty \int_0^t \frac{\xi \sin(y\xi)}{(1+\lambda_3\Phi+\lambda_3\nu\xi^2)}\ (t-z)^{p-1} \times$$

$$\left[\frac{\lambda_3(\Phi+\nu\xi^2)e^{\left(\frac{\Phi+B+\nu\xi^2}{1+\lambda_3\Phi+\lambda_3\nu\xi^2}\right)z}}{\left(\frac{\Phi+B+\nu\xi^2}{1+\lambda_3\Phi+\lambda_3\nu\xi^2}-(\Phi+B+\nu\xi^2)\right)} - \frac{\lambda_3(\Phi+\nu\xi^2)e^{(-\Phi-B-\nu\xi^2)z}}{\left(\frac{\Phi+B+\nu\xi^2}{1+\lambda_3\Phi+\lambda_3\nu\xi^2}-(\Phi+B+\nu\xi^2)\right)}\right] d\xi\ dz,$$

(35)

and corresponding shear stress

$$\tau_{SG}(y,t) = \tau_N(y,t) + \frac{2\,U\,H(t)\,\mu\,\nu\,p}{\pi}\int_0^\infty \int_0^t \frac{\xi^2 \cos(y\xi)}{(\Phi+B+\nu\xi^2)}\ (t-z)^{p-1} \times \left[\left(\frac{\Phi+B+\nu\xi^2}{1+\lambda_3\Phi+\lambda_3\nu\xi^2}\right)\lambda_3(2-\lambda_3)e^{-z} - e^{\left(\frac{\Phi+B+\nu\xi^2}{1+\lambda_3\Phi+\lambda_3\nu\xi^2}\right)z}\right] d\xi\ dz.$$

(36)

Also, solutions for Newtonian fluid when $\lambda_1 = 0$ in Eqs. 33 and 34 can be easily be retrieved in similar manners which are known in literature (Jamil, 2012).

6. Concluding remarks

In this portion is analyzed for the significance of permeability (porosity), magnetism and several rheological parameters, material parameters on the fluid flow due to accelerating plate for magnetohydrodynamics generalized Burger's fluid embedded with porous medium as a sum of Newtonian and non-Newtonian forms. In order to illustrate the differences and similarities among various graphs for relevant physical aspects, we have used different numerical values for instance, time, permeability, porosity, magnetic parameter, viscosity, non-zero constant, density, kinematic viscosity, relaxation time, retardation phenomenon, material and rheological parameters etc. However the major findings/outcomes are listed below:

1. The general solutions for velocity field and shear stress have been expressed into compact form i-e in terms of series form satisfying initial, boundary and natural conditions as well. These solutions are obtained employing four translations of integral transforms which are (i) Fourier Sine transform, (ii) Laplace transform, (iii) Inverse Fourier Sine transform and (iv) Inverse Laplace transform. The translation from (i) to (iv) are applied according to governing partial differential equation and usual initial and boundary conditions.
2. The influence of time parameter t is depicted in Fig. 1 in which as time increases the both velocity and shear stress profiles are decreasing function in absolute values.
3. Fig. 2 is plotted for the rheology of viscosity ν of the fluid in which the elastic behavior of fluid has a tendency to enhance the profiles of velocity and shear stress generated by motion of accelerating plate.
4. Figs. 3, 4, 5, and 6 have been drawn for showing the influence on material parameters ($\lambda_1, \lambda_2, \lambda_3$

and λ_4), by considering nearer and smaller values of λ_1 and λ_2 have similar and identical behavior of fluid flow as expected. It is worth pointed out that λ_1 and λ_3 relaxation and retardation phenomenon respectively have quietly contradictory effects on fluid flow for both profiles velocity as well as shear stress.

5. Fig. 7 is presents the profile of velocity and shear stress in which various extreme small values are taken for magnetic field B, it is clearly seen that it is a Lorentz force which resist the fluid flow. This is due to the fact that magnetic field B is applied in transverse direction.

6. The effects of permeability K and porosity Φ on the fluid motion are depicted in Fig. 8 which has qualitatively scattering behavior on the fluid motion.

7. Fig. 9 is drawn to give variations for the behavior on the fluid motion in presence of magnetic field as well as porosity. In which the velocity field has squeezed motion of fluid as compared with profile of shear stress.

8. Figs. 10 and 11 display the variation in presence of porosity and magnetic field respectively for six models namely (i) Generalized Burger model, (ii) Burger model, (iii) Oldroyd-B model, (iv) Maxwell model, (v) Second Grade model and (vi) Newtonian model, in which generalized Burger's model is swiftest and Newtonian model has slowest behavior on motion either the presence of porosity and magnetic field or not.
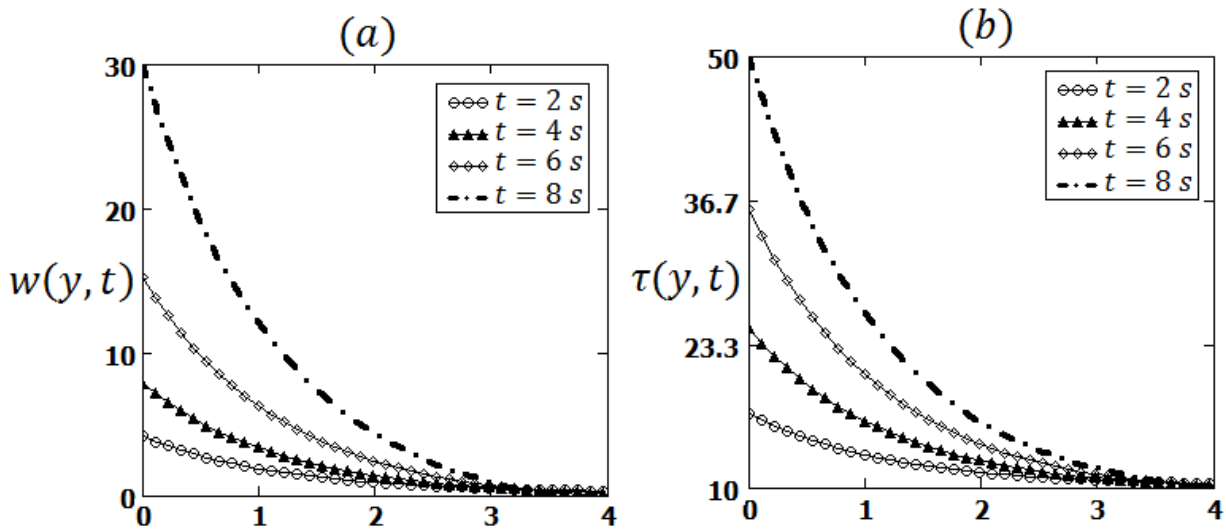

Fig. 1: Profile of velocity fields and shear stress for t at $U = 2, v = 1, \mu = 2.5, \lambda_1 = 2, \lambda_2 = 7, \lambda_3 = 5, \lambda_4 = 4, p = 2, B = 1.5, \Phi = 2.74$
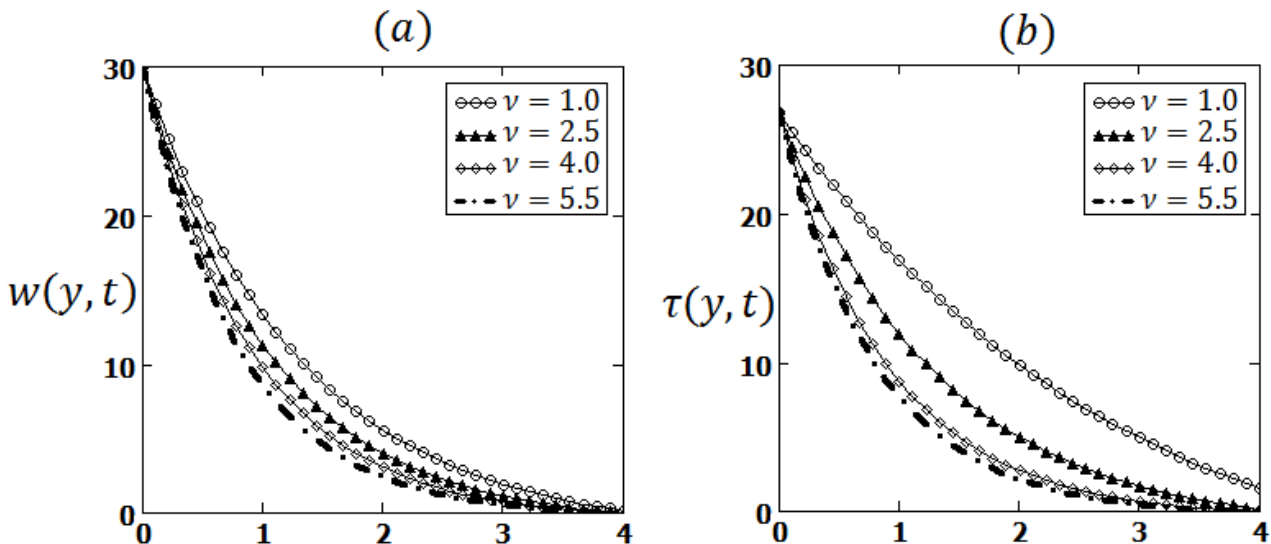

Fig. 2: Profile of velocity fields and shear stress for v at $U = 1, t = 3S, \mu = 1.3, \lambda_1 = 0.2, \lambda_2 = 0.25, \lambda_3 = 1.5, \lambda_4 = 2.25, p = 1, B = 2, \Phi = 16$
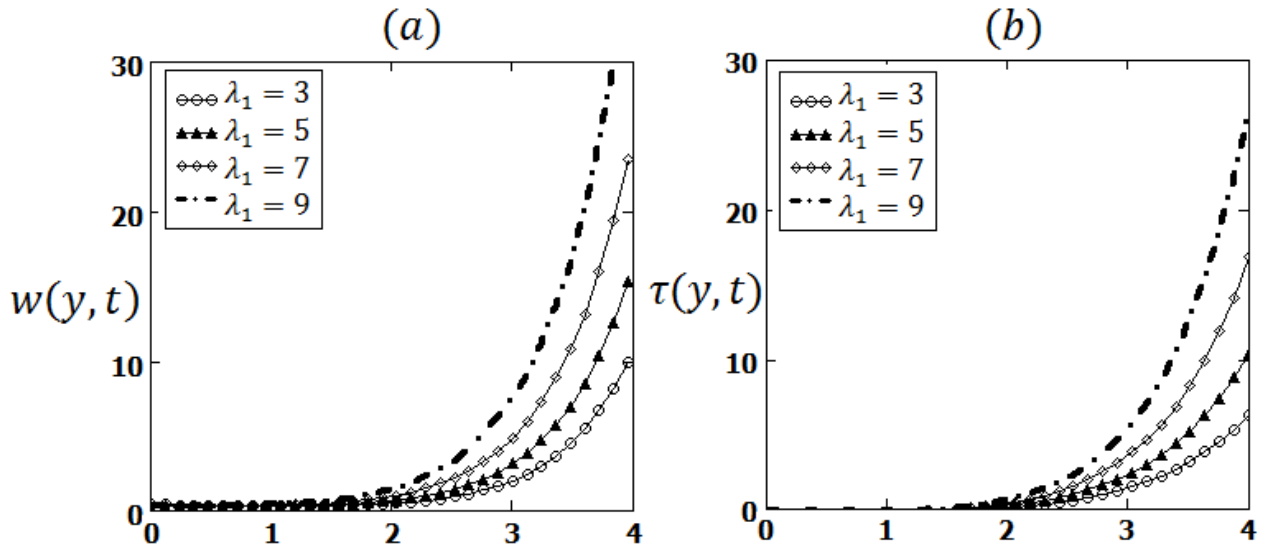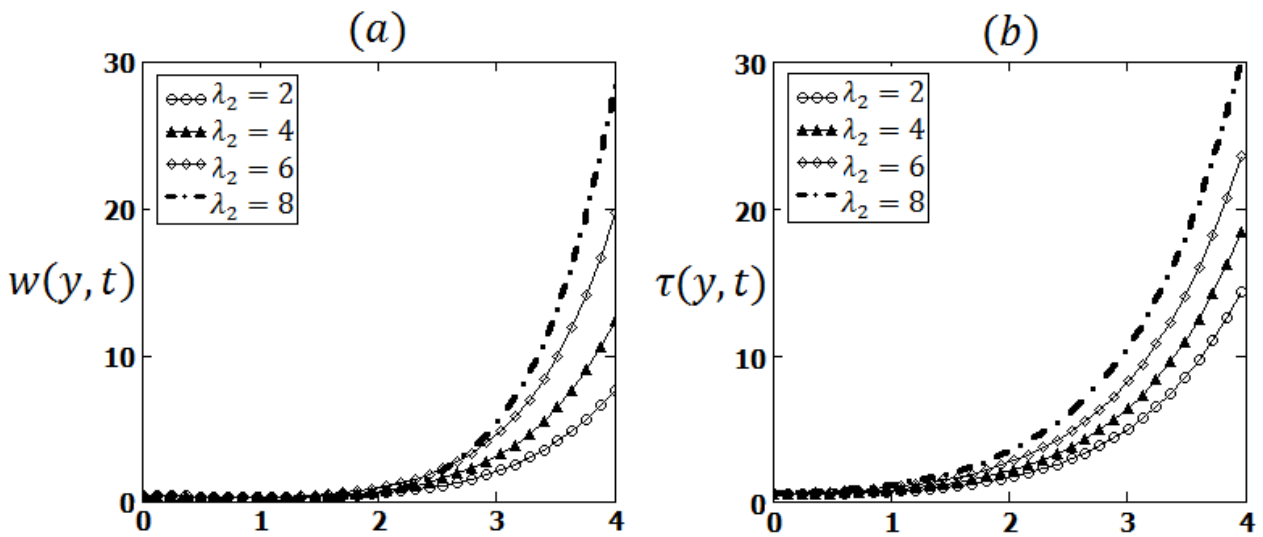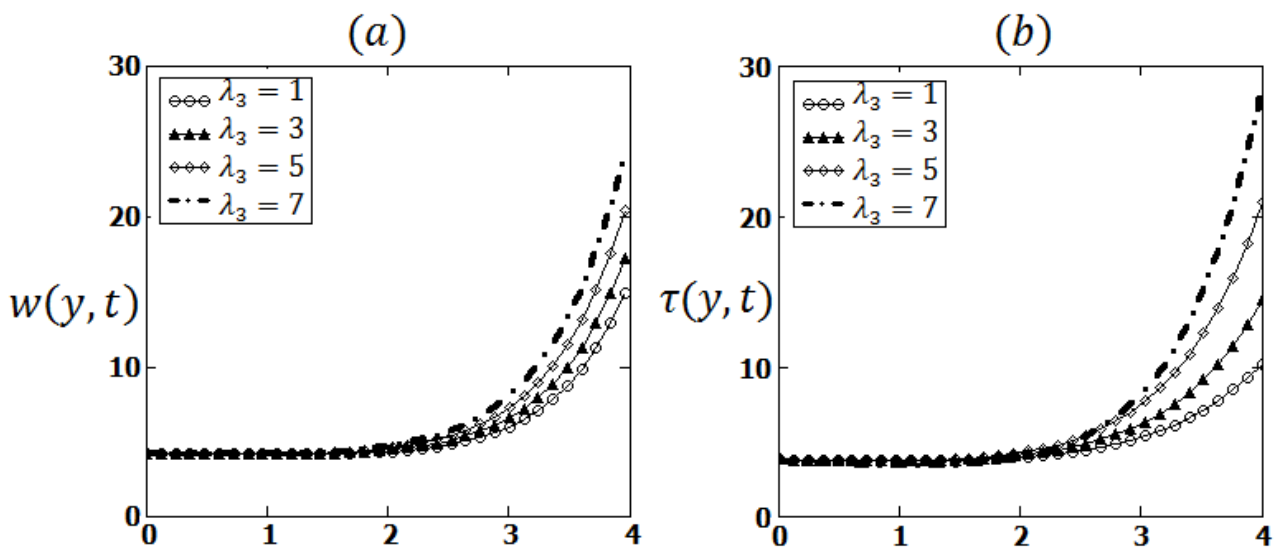
Fig. 3: Profile of velocity fields and shear stress for λ_1 at $U = 1$, $t = 3S$, $v = 3$, $\mu = 1.3$, $\lambda_2 = 0.25$, $\lambda_3 = 1.5$, $\lambda_4 = 2.25$, $p = 1$, $B = 2$, $\Phi = 16$

Fig. 4: Profile of velocity fields and shear stress for λ_2 at $U = 2$, $t = 2S$, $v = 2$, $\mu = 2.9$, $\lambda_1 = 0.3$, $\lambda_3 = 1.2$, $\lambda_4 = 0.2$, $p = 1$, $B = 0.4$, $\Phi = 8$

Fig. 5: Profile of velocity fields and shear stress for λ_3 at $U = 2$, $t = 2S$, $v = 1.5$, $\mu = 7.2$, $\lambda_1 = 5$, $\lambda_2 = 1.2$, $\lambda_4 = 19$, $p = 1$, $B = 5$, $\Phi = 3$
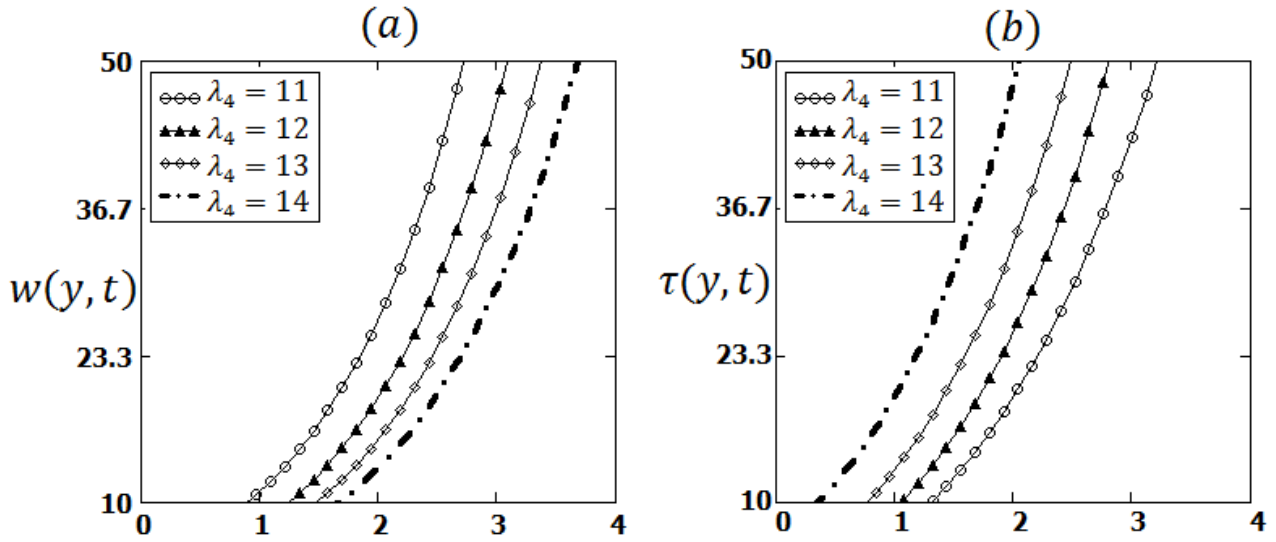
Fig. 6: Profile of velocity fields and shear stress for λ_4 at $U = 5$, $t = 3S$, $\nu = 6.25$, $\mu = 7.2$, $\lambda_1 = 12$, $\lambda_2 = 33$, $\lambda_3 = 104$, $p = 1$, $B = 0.4$, $\Phi = 7.5$
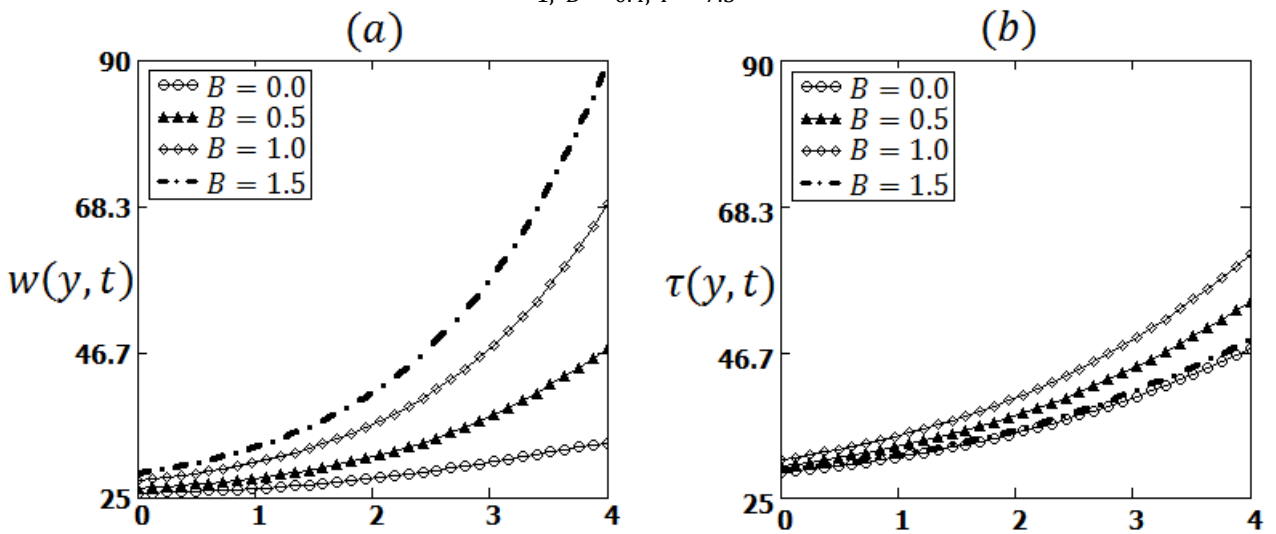

Fig. 7: Profile of velocity fields and shear stress for B at $U = 1$, $t = 2S$, $\nu = 0.25$, $\mu = 2.8$, $\lambda_1 = 1.2$, $\lambda_2 = 2.2$, $\lambda_3 = 0.5$, $\lambda_4 = 0.2$, $p = 1$, $\Phi = 3.75$
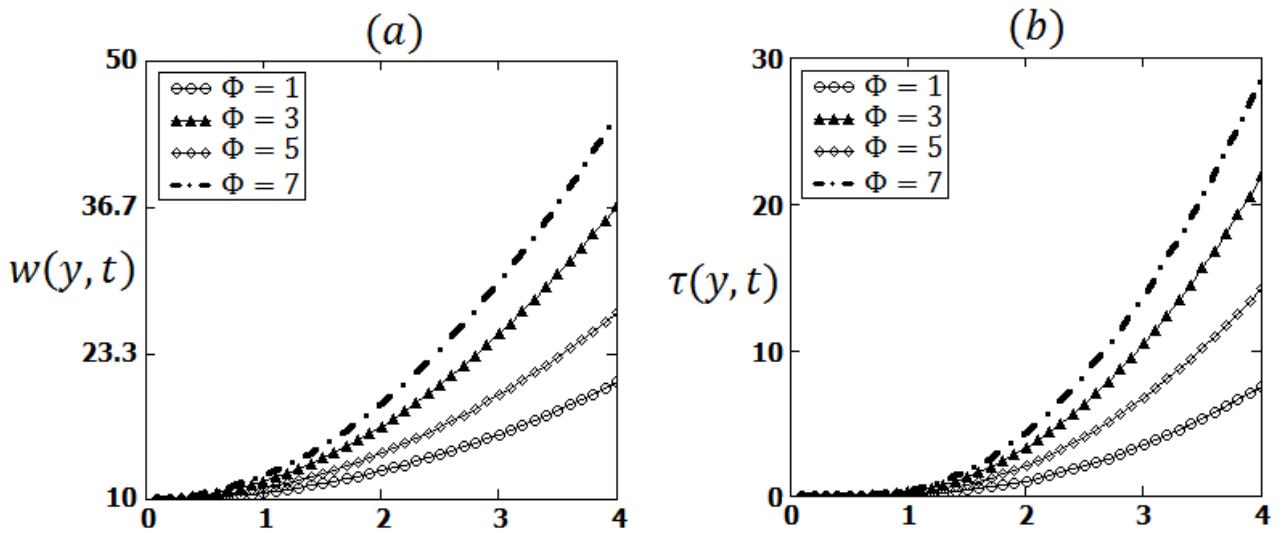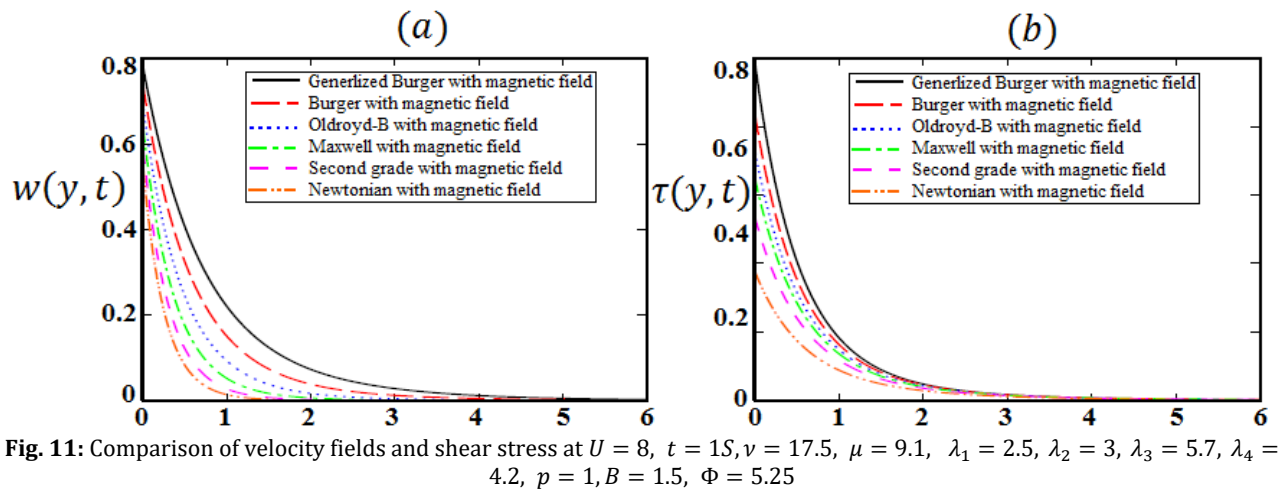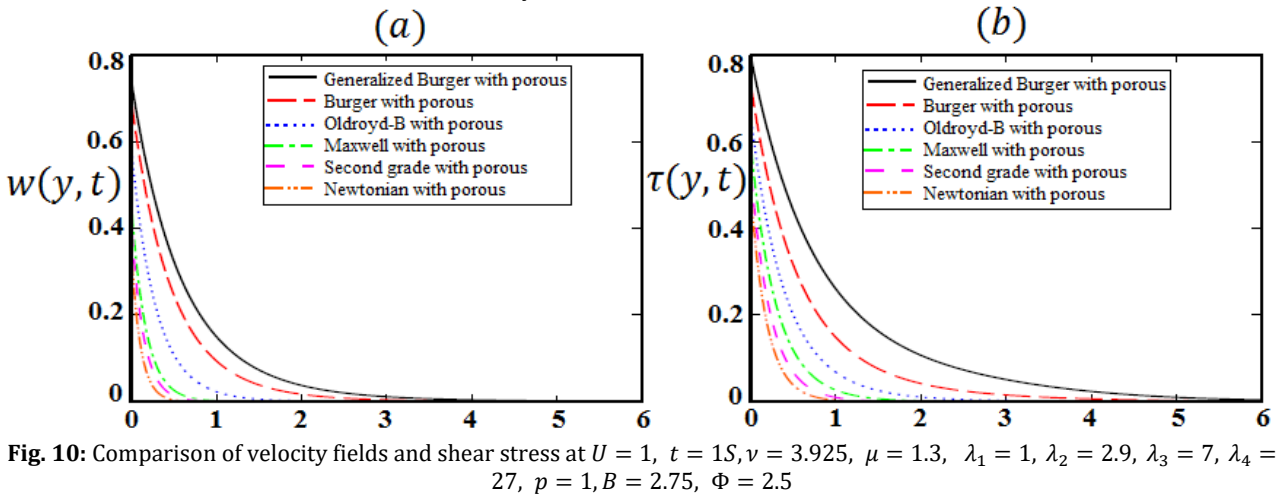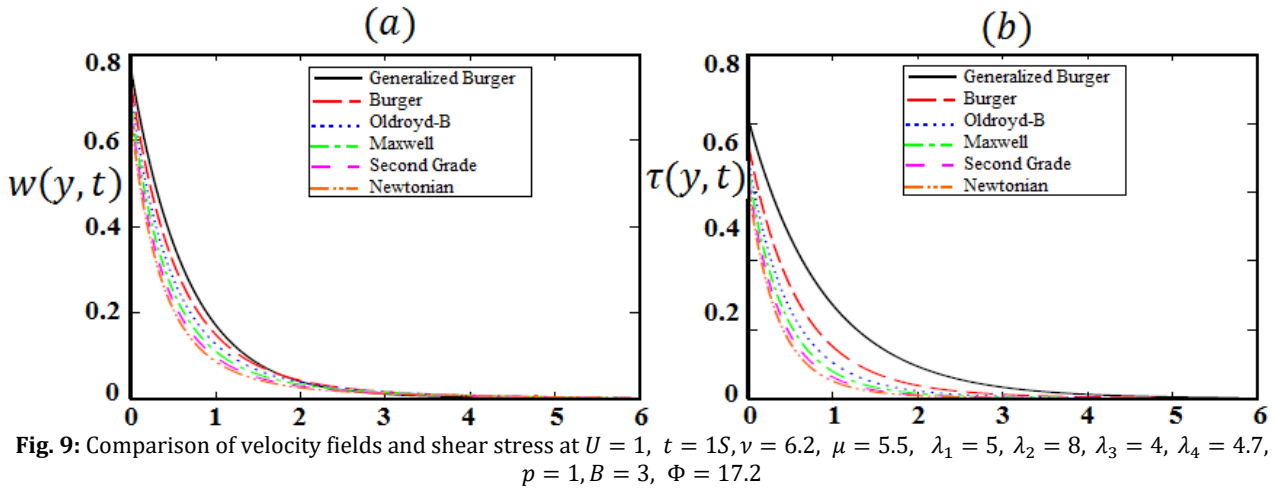

Fig. 8: Profile of velocity fields and shear stress for Φ at $U = 1$, $t = 2S$, $\nu = 1.7$, $\mu = 11.6$, $\lambda_1 = 3$, $\lambda_2 = 3.82$, $\lambda_3 = 2.8$, $\lambda_4 = 2.1$, $p = 1$, $B = 5.9$

Fig. 9: Comparison of velocity fields and shear stress at $U = 1$, $t = 1S, \nu = 6.2$, $\mu = 5.5$, $\lambda_1 = 5$, $\lambda_2 = 8$, $\lambda_3 = 4$, $\lambda_4 = 4.7$, $p = 1, B = 3$, $\Phi = 17.2$

Fig. 10: Comparison of velocity fields and shear stress at $U = 1$, $t = 1S, \nu = 3.925$, $\mu = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 2.9$, $\lambda_3 = 7$, $\lambda_4 = 27$, $p = 1, B = 2.75$, $\Phi = 2.5$

Fig. 11: Comparison of velocity fields and shear stress at $U = 8$, $t = 1S, \nu = 17.5$, $\mu = 9.1$, $\lambda_1 = 2.5$, $\lambda_2 = 3$, $\lambda_3 = 5.7$, $\lambda_4 = 4.2$, $p = 1, B = 1.5$, $\Phi = 5.25$

Competing interests

There is no conflict of interests with any person/organization upon the acceptance of this paper.

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